

GLOBAL SOLUTIONS FOR THE ZERO-ENERGY NOVIKOV-VESELOV EQUATION BY INVERSE SCATTERING

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ABSTRACT. Using the inverse scattering method, we construct global solutions to the Novikov-Veselov equation for real-valued decaying initial data q_0 with the property that the associated Schrödinger operator $-\bar{\partial}_x \partial_x + q_0$ is nonnegative. Such initial data are either critical (an arbitrarily small perturbation of the potential makes the operator nonpositive) or subcritical (sufficiently small perturbations of the potential preserve non-negativity of the operator). Previously, Lassas, Mueller, Siltanen and Stahel proved global existence for critical potentials, also called potentials of conductivity type. We extend their results to include the much larger class of subcritical potentials. We show that the subcritical potentials form an open set and that the critical potentials form the nowhere dense boundary of this open set. Our analysis draws on previous work of the first author and on ideas of P. G. Grinevich and S. V. Manakov.

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1. INTRODUCTION

The Novikov-Veselov equation at zero energy is a $(2+1)$ -dimensional completely integrable system given by

$$(1.1) \quad \begin{aligned} \partial_\tau q &= \bar{\partial}_x^3 q + \partial_x^3 q - 3\bar{\partial}_x(\bar{u}q) - 3\partial_x(uq) \\ \bar{\partial}_x u &= \partial_x q, \end{aligned}$$

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where

$$\bar{\partial}_x = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right), \quad \partial_x = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right).$$

This equation is part of a family of equations, parameterized by the energy E , studied by A. P. Veselov and S. P. Novikov [28] because of its connection with isospectral flows for the Schrödinger operator at fixed energy E (in our equation, $E = 0$). In their original work, Veselov and Novikov considered periodic solutions; P. G. Grinevich, S. V. Manakov, R. G. Novikov, and S. P. Novikov developed the inverse scattering method for nonzero energy E and initial data vanishing at infinity [9, 10, 11, 12]. We refer the reader to Kazeykina [14] for a review of results and results on long-time asymptotics at nonzero energy, and to [6] for a comprehensive review of the literature and for a self-contained introduction to the inverse scattering method for the NV equation at zero energy.

To study two-dimensional inverse scattering problems, Faddeev [8] introduced the complex geometric optics (CGO) solutions. In our case, these are a family of solutions to the zero-energy Schrödinger equation with spatial asymptotic behavior parameterized by a complex parameter k . Throughout this paper we will treat $x = (x_1, x_2) \in \mathbb{R}^2$ as a complex number $(x_1 + ix_2)$ when multiplying so that $kx = (k_1 + ik_2)(x_1 + ix_2)$. Assuming $q \in L^p(\mathbb{R}^2)$ for a fixed $p \in (1, 2)$, we say that $\psi(x, k)$ is a CGO solution for $k \in \mathbb{C} \setminus \{0\}$ if

$$\begin{aligned} -\partial_x \bar{\partial}_x \psi + q(x)\psi &= 0 \\ e^{-ikx} \psi(x, k) - 1 &\in W^{1, \tilde{p}}(\mathbb{R}^2) \end{aligned}$$

for some $p \in (1, 2)$, where \tilde{p} is the Sobolev conjugate:

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}.$$

It is more convenient to work with the normalized solutions

$$\mu(x, k) = \exp(-ikx) \psi(x, k)$$

which satisfy

$$(1.2) \quad \bar{\partial}_x (\partial_x + ik) \mu(x, k) = q(x) \mu(x, k)$$

where

$$\mu(\cdot, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2).$$

If, for some k , equation (1.2) admits a solution $h(x)$ with $h(x) \in W^{1, \tilde{p}}(\mathbb{R}^2)$, uniqueness fails and k is called an *exceptional point*. Below we will describe a spectral condition on $q(x)$ which rules out exceptional points.

The associated scattering data \mathbf{t} for the potential q is

$$(1.3) \quad \mathbf{t}(k) = \int_{\mathbb{R}^2} e_k(x) q(x) \mu(x, k) dm(x)$$

where $dm(\cdot)$ denotes Lebesgue measure on \mathbb{R}^2 and

$$e_k(x) = \exp i(kx + \bar{k}\bar{x}).$$

The map $\mathcal{T} : q \rightarrow \mathbf{t}$ defined by (1.2) and (1.3) is the *direct scattering map*.

The normalized CGO solution $\mu(x, k)$ can be recovered through the scattering data, and the potential can in turn be recovered from the reconstructed solutions $\mu(x, k)$. Indeed, if

$$\mathbf{s}(k) = \frac{\mathbf{t}(k)}{\pi \bar{k}},$$

the function $\mu(x, k)$ satisfies the $\bar{\partial}$ -equation

$$(1.4) \quad \begin{aligned} (\bar{\partial}_k \mu)(x, k) &= e_{-x}(k) \mathbf{s}(k) \overline{\mu(x, k)} \\ \mu(x, \cdot) - 1 &\in L^{\tilde{r}}(\mathbb{C}) \end{aligned}$$

where $r \in (p, 2)$ and \tilde{r} is the Sobolev conjugate of r . The potential $q(x)$ is recovered in turn using the reconstruction formula

$$(1.5) \quad q(x) = \frac{i}{\pi} \bar{\partial}_x \left(\int_{\mathbb{C}} e_{-x}(k) \mathbf{s}(k) \overline{\mu(x, k)} dm(k) \right).$$

The map $\mathcal{Q} : \mathbf{t} \rightarrow q$ defined by (1.4) and (1.5) is the *inverse scattering map*.

Formal computations [3] (see also [6, §3.6]) show that, if $q(x, \tau)$ solves NV, then the function $\mathbf{t}(k, \tau) = \mathcal{T}(q(\cdot, \tau))(k)$ should evolve by the linear evolution

$$(1.6) \quad \mathbf{t}(k, \tau) = \exp \left(i\tau \left(k^3 + \bar{k}^3 \right) \right) \mathbf{t}(k, 0).$$

Putting these ingredients together—the direct scattering transform \mathcal{T} , the linearization (1.6), and the inverse transform \mathcal{Q} —we get the solution formula

$$(1.7) \quad q(x, \tau) = \mathcal{Q} \left[\exp(i((\cdot)^3 + \overline{(\cdot)}^3)\tau) (\mathcal{T}q_0)(\cdot) \right] (x).$$

From (1.4) and (1.6), it follows that, to solve the NV equation, we need to solve the $\bar{\partial}$ -problem with parameters

$$(1.8) \quad \begin{aligned} (\bar{\partial}_k \mu)(x, k, \tau) &= e_{-x}(k) \mathbf{s}(k, \tau) \overline{\mu(x, k, \tau)} \\ \mu(x, \cdot, \tau) - 1 &\in L^r(\mathbb{C}) \end{aligned}$$

and recover

$$(1.9) \quad q(x, \tau) = \frac{i}{\pi} \bar{\partial}_x \left(\int_{\mathbb{C}} e_{-x}(k) \mathbf{s}(k, \tau) \overline{\mu(x, k, \tau)} dm(k) \right)$$

where we define

$$\mathbf{s}(k, \tau) = \frac{\mathbf{t}(k, \tau)}{\pi \bar{k}}.$$

As we will show, global existence for the formal solution (1.7) within a class of real-valued, regular, and decaying initial data is guaranteed by a simple spectral condition on the initial data. For a real-valued function $q \in L^p_{\text{loc}}(\mathbb{R}^2)$ for some $p > 1$, we say that the operator $L = -\bar{\partial}_x \partial_x + q$ is nonnegative if, for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$(1.10) \quad \int_{\mathbb{R}^2} |(\partial_x \varphi)(x)|^2 + q(x) |\varphi(x)|^2 dm(x) \geq 0.$$

Murata [18] defined a trichotomy of potentials for the zero-energy Schrödinger operators (see Simon [25] for an earlier and closely related trichotomy in the study of Schrödinger semigroups). This trichotomy plays a crucial role in our analysis and may distinguish distinct dynamical regimes for the NV equation.

Definition 1.1. Fix $p > 1$. A potential $q \in L^p_{\text{loc}}(\mathbb{R}^2)$ is

- (i) *subcritical* if there is a positive Green's function for the operator L ,
- (ii) *critical* if there is no positive Green's function but $L \geq 0$, and
- (iii) *supercritical* if $-\bar{\partial}_x \partial_x + q$ is not nonnegative.

Murata (see [18, Definition 2.1] and accompanying discussion) shows these are the only three possibilities, so the condition (1.10) implies that q is either critical or subcritical. Theorem 2.12 of [7] implies that if q is uniformly locally in $L^p(\mathbb{R}^2)$ for $p > 1$, then $-\Delta + q \geq 0$ if and only if there exists a positive distributional solution to the Schrödinger equation (we use here the fact that, for $p > 1$, the uniformly locally L^p potentials are contained in Kato class K_2 considered in [7]; see [7, §1.2, Remark 1(b)] and see [7, §2.5] for further discussion and references to the original work of Allegretto, Moss, and Piepenbrink). Additionally, if $q(x)$ has sufficient decay, the equation $-\bar{\partial}_x \partial_x \psi + q\psi = 0$ admits a positive, *bounded*, distributional solution ψ in the critical case, and a positive solution with *logarithmic growth* in the subcritical case [18, Theorem 5.6]. In Section 3 we show that subcritical potentials are an open set in the topology $L^p_\rho(\mathbb{R}^2)$ with $p \in (1, 2)$ and $\rho > 1 + 2/\tilde{p}$ and the critical potentials are the boundary of this set.

The asymptotic behavior as $|x| \rightarrow \infty$ of the positive solution is closely related, via a Fourier-like duality, to the asymptotic behavior as $|k| \downarrow 0$ of the scattering transform $\mathbf{t}(k)$. Indeed, Nachman [21, Theorem 3.3] showed that, for critical potentials, $|\mathbf{t}(k)| \leq C|k|^\varepsilon$ for some $\varepsilon > 0$ and small k . This means that $\mathbf{s}(k) = \mathcal{O}(|k|^{\varepsilon-1})$ is in $L^p_{\text{loc}}(\mathbb{C})$ for any $p \in (1, \infty)$ and the integral operator (2.8) for the problem (1.8) has range in L^∞ functions using (2.3) of Lemma 2.1. By contrast, Music [19] showed that, for subcritical potentials,

$$\mathbf{s}(k) \underset{|k| \rightarrow 0}{\sim} -\frac{1}{\bar{k} \log(|k|^2)} \left(1 + \mathcal{O}\left((- \log(|k|^2))^{-1}\right)\right)$$

and the same key integral operator does not have range in bounded functions. This singularity is the cause of some technical pain in our computations but, thanks to a density argument (see Lemma 4.5 and Corollary 5.3), does not pose essential problems. In Proposition 5.4 we prove that any scattering transform that has this singularity and also has certain decay properties (see Definition 2.7) must come from a critical or subcritical potential.

To state the theorem, denote by $L^p_\rho(\mathbb{R}^2)$ the space

$$\left\{ f \in L^p(\mathbb{R}^2) : (1 + |\cdot|^2)^{\rho/2} f(\cdot) \in L^p(\mathbb{R}^2) \right\}$$

and by $W^{n,p}_\rho(\mathbb{R}^2)$ the space of measurable functions with n weak derivatives in $L^p_\rho(\mathbb{R}^2)$.

Theorem 1.2. *Let $q_0 \in W^{5,p}_\rho(\mathbb{R}^2)$ for $p \in (1, 2)$ and $\rho > 1$ be a critical or subcritical potential. Then $q(x, \tau)$ given by (1.7) is a global-in-time, classical solution of NV (1.1) with $\lim_{\tau \rightarrow 0} q(x, \tau) = q_0(x)$ pointwise.*

The regularity assumption guarantees that the inverse scattering method will yield a classical solution of NV. By contrast, Angelopoulos [1] recently proved local well-posedness for NV in the space $H^s(\mathbb{R}^2)$ for any $s > \frac{1}{2}$. The inverse scattering method captures global properties of the solution at the expense of more stringent regularity and decay assumptions on the initial data.

The condition that the potential is subcritical or critical is necessary in the above theorem using our methods. This is because in [20], Music, Perry, and Siltanen construct radial, compactly supported, supercritical potentials q_0 where the scattering transform $\mathbf{t}(k)$ has a circle of singularities and the current formulation of the inverse scattering method fails. P.G. Grinevich and R.G. Novikov [13] give explicit examples of point potentials with similar contour-type singularities. Taimanov and Tsarev (see [27] and references therein) construct examples of supercritical potentials that give rise to solutions of the zero-energy NV equation that blow up in finite time.

The technical core of our result is a careful analysis of the $\overline{\partial}$ -problem (1.8) with parameters x, τ . The first technical challenge is that the scattering data $\mathbf{t}(k)$ for our class of potentials may be mildly singular as $|k| \downarrow 0$ (see Lemma 2.6). We deal with the singularity by proving continuity properties of the direct and inverse scattering maps and approximating $\mathbf{t}(k)$, the scattering transform of a potential q in our class by a smooth function with compact support away from $k = 0$.

The second technical challenge is that, in order to prove that the reconstructed potential (1.9) solves the NV equation, we must first prove that it is a real-valued function. We will show that this is the case by adapting ideas of Grinevich [9] and Grinevich-Novikov [12].

Lassas, Mueller, Siltanen, and Stahel [16] study the same problem for initial data of conductivity type which have very well-behaved scattering transforms as shown by Nachman in his 1996 paper [21] on Calderon's problem. These authors obtain a number smoothness and decay results for this class, but they cannot prove that the inverse scattering method yields solutions to the Novikov-Veselov equation. Combining our result with the earlier one from Lassas, Mueller, and Siltanen [15] gives global solutions to Novikov-Veselov which satisfy the decay condition $|q(x, \tau)| \leq \langle x \rangle^{-2}$ if we start with a critical potential $q(x, 0) \in C_c^\infty(\mathbb{R}^2)$. Although this special case has already been proved by Miura map methods by Perry [23], the proof given here is arguably more direct and simpler.

In Section 2.1, we state some theorems that will be used throughout the paper. In the first half, Section 2.1, we recall facts about the $\overline{\partial}$ problem, and in Section 2.2, we recall results of Music [19] on the properties of the scattering transform for critical and subcritical potentials [19]. Music's characterization of the range of \mathcal{T} will allow us to analyze the inverse map \mathcal{Q} rigorously. In Section 3, we prove that the set of subcritical potentials is open and that the set of critical potentials is its nowhere dense boundary. In Section 4, we establish continuity in q for the direct scattering map \mathcal{T} , and continuity in $\mathbf{s}(k)$ of the inverse scattering map \mathcal{Q} . In Section 5, we exploit ideas of P.G. Grinevich and S.V. Manakov [10] to prove necessary and sufficient conditions for the scattering data $\mathbf{t}(k)$ to be the scattering data of a real potential $q(x)$ repairing a gap [24] in the proof of [16, Theorem 4.1]. We apply the ideas of Grinevich-Manakov [10] directly to the zero-energy case instead of treating it as a limit of negative-energy inverse scattering (see also Grinevich [9]). With this result, we prove that the reconstructed potential $q(x, \tau)$ stays real for all time. Crucially, the reality of the inverse map is needed to show that the evolved $\mu(x, k, \tau)$ continue to solve the problem (1.2). In Section 6, we follow the method outlined in the review article of Croke, Mueller, Music, Perry, Siltanen and Stahel [6] to show that the potential solves the Novikov-Veselov equation. We discuss some open problems in section 7.

2. PRELIMINARIES

2.1. The $\bar{\partial}$ -Problem. Here we review some important facts about the $\bar{\partial}$ operator and its inverse that will be used in what follows. Further details and references can be found, for example, in [22, section 2] or [2, chapter 4]. We denote by P the integral operator

$$[Pf](z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{z - \zeta} f(\zeta) dm(\zeta)$$

where, in the integrand, $z = z_1 + iz_2$ and $\zeta = \zeta_1 + i\zeta_2$. We recall without proof the following basic estimates. In what follows, C_p (resp. $C_{p,q}$) denotes a numerical constant depending only on p (resp. p, q).

Lemma 2.1. (i) For any $p \in (2, \infty)$ and $f \in L^{2p/(p+2)}(\mathbb{C})$,

$$(2.1) \quad \|Pf\|_p \leq C_p \|f\|_{2p/(p+2)}.$$

Moreover, $\nabla Pf \in L^{2p/(p+2)}(\mathbb{C})$ and

$$(2.2) \quad \|\nabla Pf\|_{2p/(p+2)} \leq C_p \|f\|_{2p/(p+2)}.$$

(ii) For any p, q with $1 < q < 2 < p < \infty$ and any $f \in L^p(\mathbb{C}) \cap L^q(\mathbb{C})$, the estimate

$$(2.3) \quad \|Pf\|_{\infty} \leq C_{p,q} \|f\|_{L^p \cap L^q}$$

holds.

(iii) If $v \in L^s(\mathbb{C})$, $p \in (2, \infty)$ and $q \in (2, \infty)$ with $q^{-1} + 1/2 = p^{-1} + s^{-1}$, then for any $f \in L^p(\mathbb{C})$,

$$(2.4) \quad \|P(vf)\|_q \leq C_{p,q} \|v\|_s \|f\|_p.$$

The operator P is the solution operator for the problem $\bar{\partial}u = f$:

Lemma 2.2. Suppose that $p \in (2, \infty)$, that $u \in L^p(\mathbb{C})$, that $f \in L^{2p/(p+2)}(\mathbb{C})$, and that $\bar{\partial}u = f$ in distribution sense. Then $u = Pf$. Conversely, if $f \in L^{2p/(p+2)}(\mathbb{C})$ and $u = Pf$, then $\bar{\partial}u = f$ in distribution sense.

We will need the following generalized Liouville Theorem for quasi-analytic functions, due in this form to Brown and Uhlmann. Music [19] has an extension of this lemma to include certain negatively weighted L^p spaces.

Theorem 2.3 ([5, Corollary 3.11]). Suppose that $u \in L^p(\mathbb{C}) \cap L^2_{\text{loc}}(\mathbb{C})$ for some $p \in [1, \infty)$ and that

$$\bar{\partial}u = au + b\bar{u}$$

for a and b belonging to $L^2(\mathbb{C})$. Then $u \equiv 0$.

Finally, solutions of $\bar{\partial}u = f$ for rapidly decaying f have a large- z expansion.

Lemma 2.4. Suppose that $p \in (2, \infty)$, that $u \in L^p(\mathbb{C})$, that $f \in L^{2p/(p+2)}_n(\mathbb{C})$, and that $\bar{\partial}u = f$. Then

$$z^n \left[u(k) - \sum_{j=0}^{n-1} \frac{1}{z^{j+1}} \int \zeta^j f(\zeta) dm(\zeta) \right] \in L^p(\mathbb{C}).$$

We will also use the following estimates from [4] on the fundamental solution kernel for the linear problem $v_{\tau} = \partial_x^3 v + \bar{\partial}_x^3 v$ associated to the NV equation.

Lemma 2.5. *Let*

$$(2.5) \quad I_\tau(x) = \int e^{i\tau(k^3 + \bar{k}^3) - i(kx + \bar{k}\bar{x})} dm(k).$$

Then:

$$I_\tau(x) = \tau^{-2/3} I_1 \left(x\tau^{-1/3} \right)$$

and the estimates

$$\begin{aligned} |I_1(x)| &\leq C(1 + |x|)^{-1/2}, \\ |\nabla_x I_1(x)| &\leq C \end{aligned}$$

hold.

For a proof see [4, Proposition 5.4].

2.2. Properties of the scattering transform. The following is a reformulation of Theorem 1.4 and Lemma 5.2 from Music [19]. Instead of stating the lemma in terms of the space that the potential q lives, we state the assumptions that the scattering transform satisfies. This is the purpose of defining the space $\mathcal{X}_{n,r}^\epsilon$ in Definition 2.7. See Remark 2.8 for more details.

For subcritical potentials, we let $\psi(x)$ be the unique positive solution of $-\bar{\partial}_x \partial_x \psi + q\psi = 0$ such that

$$(2.6) \quad \psi(x) = a \log |x| + O(1).$$

We also set

$$(2.7) \quad c_\infty = \lim_{R \rightarrow \infty} \frac{1}{\pi R^2} \int_{|x| < R} (\psi(x) - a \log |x|) dm(x).$$

Note that for critical potentials, we have $a = 0$ and $c_\infty = 1$, and the following results are originally due to Nachman [21, Section 3] in this case.

Lemma 2.6. *Let $q \in L_\rho^p(\mathbb{R}^2)$ with $p \in (1, 2)$, and $\rho > 1$, and suppose that q is either critical or subcritical. Let $\psi(x)$ denote a positive function solving $(-\bar{\partial}_x \partial_x + q)\psi = 0$. Then $q(x)$ has no exceptional points, and the scattering transform $\mathbf{t}(k)$ of $q(x)$ has the following properties:*

(i) *For small k and some $\varepsilon > 0$, we have*

$$\mathbf{t}(k) = \frac{1}{2} \frac{\pi a}{c_\infty - a(\log |k| + \gamma)} + O(|k|^\varepsilon),$$

where c_∞, a are defined in equations (2.6) and (2.7).

(ii) *$\mathbf{t}(k)/\bar{k} \in L^r(|k| > \epsilon)$ for $r \in (\bar{p}', \infty)$ and every $\epsilon > 0$. Moreover, $\mathbf{t}(k)/\bar{k} \in C^0(\mathbb{C} \setminus 0)$.*

(iii) *If $q \in W_\rho^{n,p}(\mathbb{R}^2)$ for $n \geq 1$ then $|k|^n \mathbf{s}(k) \in L^r(\mathbb{C})$.*

In fact, we will use the above properties for scattering transforms coming from real potentials to define a class of scattering transforms.

Definition 2.7. The space $\mathcal{X}_{n,r}^\epsilon$ for $n \geq 1$, $r \in (1, 2)$, and $\epsilon > 0$ is the closure of $C_c^\infty(\mathbb{C})$ functions which satisfy the relation $\bar{k}f(k) = -kf(-\bar{k})$ in the norm

$$\|f\|_{\mathcal{X}_{n,r}^\epsilon} = \|f\|_{L^2(\mathbb{C})} + \|(\cdot)^n f(\cdot)\|_{L^{r'+\epsilon} \cap L^r(\mathbb{C})} + \|(\cdot)^n f(\cdot)\|_{L^r(\mathbb{C})}.$$

Remark 2.8. The symmetry requirement guarantees that the reconstructed potential is real-valued; see Lemmas 5.1 and 5.2 of Section 5. The conditions in this definition are preserved under the linearized NV flow (1.6). By Lemma 2.6 scattering transforms $\mathbf{s}(k)$ coming from potentials $q \in W_\rho^{n,p}(\mathbb{R}^2)$ are in $\mathcal{X}_{n,r}^\epsilon$ for all $r \in (\bar{p}', \infty)$ and $\epsilon > 0$.

These properties of $\mathbf{s}(k) \in \mathcal{X}_{n,r}^\epsilon$ are used in Music [19] in the inverse problem to reconstruct $\mu(x, k)$ and obtain a large- k expansion of $\mu(x, k)$. For the large- k expansion to exist pointwise, we need the decay $k^n \mathbf{s}(k) \in L^{2+\epsilon} \cap L^{2-\epsilon}(\mathbb{C})$ (see Lemmas 5.5 and 5.6 of [19]). We need this decay for all k away from the origin, and in this region it suffices to have $\mathbf{s} \in L^2(\mathbb{C})$ instead of $L^{2+\epsilon}(\mathbb{C})$. This is why we only need the homogeneous norm on $L^{r'+\epsilon}(\mathbb{C})$. With this norm and the embedding $\mathcal{X}_{n+1,r}^\epsilon \subset L_n^1$, we have $\mu(x, \cdot) - 1 \in L^{\bar{r}}(\mathbb{C})$ and for every x we get the expansion (2.11).

In what follows, we define the operator $T_{x,\tau}$ by

$$(2.8) \quad (T_{x,\tau} f)(k) = P_k [\mathbf{s}(\cdot, \tau) e_{-x}(\cdot) \bar{f}](k)$$

where

$$[P_k f](k) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{k - \zeta} f(\zeta) dm(\zeta).$$

We now recall from [19] that the solution of (1.8) is differentiable in the parameters x and τ .

Lemma 2.9. *If $\mathbf{s} \in \mathcal{X}_{n,r}^\epsilon$ then the unique solution $\mu(x, \cdot, \tau) - 1 \in L^{\bar{r}}(\mathbb{C})$ of equation (1.8) is α times differentiable in x and m times differentiable in t for $n \geq 3m + |\alpha|$. Additionally, the derivatives of the map $(x, \tau) \rightarrow \mu(x, \cdot, \tau) - 1 \in L^{\bar{r}}(\mathbb{C})$ satisfy $\partial_\tau^m D_x^\alpha \mu(x, \cdot, \tau) \in L^{\bar{r}}(\mathbb{C})$. The derivatives are given by*

$$(2.9) \quad \partial_\tau^m D_x^\alpha \mu(x, k, \tau) = [I - T_{x,\tau}]^{-1} P_k f(x, \cdot, \tau)$$

where

$$(2.10) \quad f(x, k, \tau) = [\partial_\tau^m D_x^\alpha \mathbf{s}(k, \tau) e_{-x}(k)] \overline{\mu(x, k, \tau)},$$

and $P_k [\mathbf{s}(k, \tau) e_{-x}(k) f(x, k, \tau)] \in L^{\bar{r}}(\mathbb{C})$.

Finally, it follows from Lemma 2.4 that $\mu(x, k, \tau)$ has a large- k expansion. We refer the reader to [19] for a full proof.

Lemma 2.10. *If $\mathbf{s} \in \mathcal{X}_{n,r}^\epsilon$ and if μ solves the equation (1.8) with $\mu(x, \cdot) - 1 \in L^{\bar{r}}(\mathbb{C})$, then μ admits the large- k expansion*

$$(2.11) \quad \mu(x, k, \tau) = 1 + \sum_{j=1}^n \frac{a_j(x, \tau)}{k^j} + o(|k|^{-n})$$

for fixed x and τ . Moreover, we may take α spatial derivatives and m time derivatives with $n \geq |\alpha| + 3m$ to get

$$\partial_\tau^m D_x^\alpha (\mu(x, k, \tau) - 1) = \sum_{j=1}^{n-|\alpha|-3m} \frac{\partial_\tau^m D_x^\alpha a_j(x, \tau)}{k^j} + o(|k|^{-n+|\alpha|+3m}).$$

We will use the above lemma to take up to four spatial derivatives, or one time and one space derivative of $a_1(x, \tau)$. This requires $\mathbf{t}(k) \in \mathcal{X}_{5,r}^\epsilon$ and is why we assume that $q(x, 0) \in W_\rho^{5,p}(\mathbb{R}^2)$. It is also important to notice that these are not weak derivatives as $-i\bar{\partial}^{-1} q = a_1(x, \tau)$ and $W_\rho^{5,p}(\mathbb{R}^2) \subset C^3(\mathbb{R}^2)$.

3. TOPOLOGY OF THE SET OF SUBCRITICAL POTENTIALS

The set of subcritical potentials is actually bigger than the set of critical potentials. Murata [18, Theorem 2.4] shows that the set of critical potentials is not stable under compact perturbations. Here we go a step further and show the subcritical potentials are an open set in the topology of $L^p_\rho(\mathbb{R}^2)$ for certain p and ρ . In the corollary to the theorem we show that subcritical potentials are open in the $W^{1,p}_\rho(\mathbb{R}^2)$ topology for the same p and ρ as in Theorem 1.2.

Theorem 3.1. *The set of subcritical potentials is open in the $L^{p_1}_{\rho_1}(\mathbb{R}^2)$ topology for $p_1 \in (1, 2)$ and $\rho_1 > 1 + 2/\tilde{p}$ and the set of critical potentials is its boundary.*

Proof. We prove that the set of subcritical potentials is open by finding positive logarithmically growing solutions, ϕ_v , to $(-\Delta + v)\phi_v = 0$ for all v close enough to a given subcritical potential q .

Let $G_0(x) = -1/(2\pi) \log|x|$. Then, by Nachman [21, Lemma 3.2] we have for $f \in L^{p_2}_{\rho_2}(\mathbb{R}^2)$ with $\rho_2 > 1$ and $p_2 \in (1, 2)$

$$\left\| G_0 * f + \frac{1}{2\pi} \log|x| \int f dm(x) \right\|_{L^{\tilde{p}_2}} \leq c \|f\|_{L^{p_2}_{\rho_2}}$$

and

$$\|\nabla G_0 * f\|_{L^{\tilde{p}_2}} \leq c \|f\|_{L^{p_2}}.$$

Modifying both of these slightly, we let

$$Tf = G_0 * f + \frac{1}{2\pi} \log(|x| + e) \int f dm(x)$$

and get

$$\|Tf\|_{W^{1,\tilde{p}_2}} \leq c \|f\|_{L^{p_2}_{\rho_2}}.$$

In particular, by Sobolev embedding we have

$$\|Tf\|_{L^\infty} \leq \|f\|_{L^{p_2}_{\rho_2}}.$$

Let ϕ_q be a positive solution to the Schrodinger equation and let c_∞ be given by (2.7). By Nachman [21, Lemma 3.5] for critical potentials and Music [19, Theorem 1.4] for subcritical potentials with $c_\infty \neq 0$, we have that $[I + G_0 * (q \cdot)]$ is invertible for critical/subcritical q on $W^{1,\tilde{p}}_{-\beta}(\mathbb{R}^2)$ for $\beta > 2/\tilde{p}$. If $c_\infty = 0$, we may use the same argument from Music [19, Theorem 1.4 and Lemma 4.1] to scale q , and therefore ϕ_q , so that $c_\infty \neq 0$ and we may reduce to the case. So for some $\epsilon > 0$ the operator $[I + G_0 * (v \cdot)]$ is invertible for all potentials v satisfying $\|q - v\|_{L^p_\rho} < \epsilon$. Thus we have $W^{1,\tilde{p}}_{-\beta}$ distributional solutions to $(-\Delta + v)\phi_v = 0$ by taking $\phi_v = [I + G_0 * (v \cdot)]^{-1} c_\infty$. Because q is subcritical $\phi_q = a \log|x| + O(1)$ is a positive solution. The solutions ϕ_v satisfy the equation

$$\phi_v = c_\infty - G_0 * (v\phi_v).$$

Taking the difference between these two solutions we find

$$\begin{aligned} \phi_q - \phi_v &= G_0 * (v\phi_v - q\phi_q) + \frac{1}{2\pi} \log(|x| + e) \int (v\phi_v - q\phi_q) dm(x) \\ &\quad - \frac{1}{2\pi} \log|x| \int (v\phi_v - q\phi_q) dm(x) \\ &= T(v\phi_v - q\phi_q) - \frac{1}{2\pi} \log(|x| + e) \int (v\phi_v - q\phi_q) dm(x) \end{aligned}$$

which becomes

$$|\phi_q - \phi_v| \leq c \|v\phi_v - q\phi_q\|_{L_{\rho_2}^{p_2}} + \frac{1}{2\pi} \log(|x| + e) \|v\phi_v - q\phi_q\|_{L^1}$$

for $p_2 \in (1, 2)$ and $\rho_2 > 1$. We may choose these constants by taking β close enough to $2/\tilde{p}$ so that $\rho_2 = \rho_1 - \beta > 1$. Also, we have the embedding $\|f\|_{L^1} \leq c\|f\|_{L_{\rho_2}^{p_2}}$ so

$$\phi_v \geq \phi_q - c \|v\phi_v - q\phi_q\|_{L_{\rho_2}^{p_2}} - \frac{1}{2\pi} \log(|x| + e) \|v\phi_v - q\phi_q\|_{L_{\rho_2}^{p_2}}.$$

The right hand side is positive and logarithmically growing for $\|v\phi_v - q\phi_q\|_{L_{\rho_2}^{p_2}}$ small. The norm is small because $\|\phi_v - \phi_q\|_{W_{-\beta}^{1,\tilde{p}}} \leq c\|q - v\|_{L_{\rho}^p}$ by the second resolvent formula applied to the operators $[I + G_0 * (v \cdot)]$. The function ϕ_v is a positive distributional solution to $(-\Delta + v)\phi_v = 0$ so by [7, Theorem 2.12] v is critical or subcritical. It follows from Murata [18, Theorem 5.6] that v is subcritical because the positive solutions for critical potentials in this weighted space have the asymptotics $\phi = c + o(1)$ whereas the positive solutions for subcritical potentials obey $\phi = a \log|x| + O(1)$.

In Theorem 2.4 of the same paper, Murata showed that nonnegative compact perturbations of critical potentials are subcritical and nonpositive perturbations are supercritical. Thus critical potentials are a subset of the boundary of subcritical potentials. Looking at the form (1.10) we see that limits of subcritical potentials must either be subcritical or critical. Therefore, critical potentials form the entire boundary. \square

Now we apply the above lemma to the set of potentials under consideration in Theorem 1.2.

Corollary 3.2. *The set of subcritical potentials is open in the $W_{\rho}^{1,p}(\mathbb{R}^2)$ topology for $p \in (1, 2)$ and $\rho > 1$ and the set of critical potentials is its boundary.*

Proof. Be Sobolev embedding $W_{\rho}^{1,p}(\mathbb{R}^2) \subset L_{\rho}^s(\mathbb{R}^2)$ for all $s \in (p, \tilde{p})$. In particular we may choose $s < 2$ but close enough to 2 so that $\rho > 1 + 2/\tilde{s}$. The result then follows from the previous theorem. \square

4. CONTINUITY OF MAPS

In this section, we prove that the scattering data, \mathbf{t} , depends continuously on the potential q and that the solution of (1.8) is continuous in $\mathbf{s}(k)$ in a suitable sense. Using the continuity of the inverse scattering map defined by the $\bar{\partial}$ -problem, we can show that our reconstructed $q(x, \tau)$ is differentiable. In fact by approximating the scattering transform $\mathbf{s}(k)$ by $\mathbf{s}_{\epsilon}(k) \in C_c^{\infty}(\mathbb{C})$ in the $\mathcal{X}_{n,r}^{\epsilon}$ norm, we approximate the solution $q(x, \tau)$ by the reconstructed $q_{\epsilon}(x, \tau)$. In this way, the results showing that each $q_{\epsilon}(x, \tau)$ solves the Novikov-Veselov equation carry over to the general subcritical case so long as the initial data q_0 has enough derivatives.

In what follows, $\mathcal{B}(X)$ denotes the Banach space of bounded operators from a Banach space X to itself with the usual norm.

4.1. Continuous dependence of \mathbf{t} on q . Recall from Nachman [21, §1] that if $p \in (1, 2)$, $u \in W^{1,\tilde{p}}(\mathbb{R}^2)$, $f \in L^p(\mathbb{R}^2)$, $k \in \mathbb{C} \setminus \{0\}$, and $\bar{\partial}_x(\partial_x + ik)u = f$ in distribution sense, then

$$u = g_k * f$$

for a convolution kernel g_k which satisfies the estimate

$$\|g_k * f\|_{W^{1,\bar{p}}} \leq c_0(k, p) \|f\|_p.$$

Let us define

$$S_k(q)f = g_k * (qf).$$

It follows from the estimates above that for $q \in L^p(\mathbb{R}^2)$,

$$(4.1) \quad \|S_k(q)\|_{\mathcal{B}(W^{1,\bar{p}})} \leq c_0(k, p) \|q\|_p$$

while $S_k(q)1 := g_k * q$ satisfies

$$(4.2) \quad \|S_k(q)1\|_{W^{1,\bar{p}}} \leq c(k, p) \|q\|_p.$$

In (4.1) we used the Sobolev inequality $\|f\|_\infty \leq c_p \|f\|_{W^{1,\bar{p}}}$.

We now consider the continuity of $\mu(x, k; q)$ defined by

$$\mu(x, k; q) - 1 = (I - S_k(q))^{-1} S_k(q)1$$

as a function of q .

Lemma 4.1. *Fix $p \in (1, 2)$, $k \in \mathbb{C} \setminus \{0\}$, and $q_0 \in L^p(\mathbb{R}^2)$. Suppose that*

$$R(q_0) := (I - S_k(q_0))^{-1}$$

exists as a bounded operator from $W^{1,\bar{p}}(\mathbb{R}^2)$ to itself. There is a number $r > 0$ and a constant $c(p, k, q_0)$ so that for all $q \in L^p(\mathbb{R}^2)$ with $\|q_0 - q\|_p \leq r$, the estimate

$$\|\mu(\cdot, k, q) - \mu(\cdot, k, q_0)\|_{W^{1,\bar{p}}} \leq c(p, k, q_0) \|q - q_0\|_p$$

holds.

Proof. In what follows c denotes a constant depending only on k , p , and q_0 whose value may vary from line to line. We will use the estimates

$$(4.3) \quad \|S_k(q) - S_k(q_0)\|_{\mathcal{B}(W^{1,\bar{p}})} \leq c \|q - q_0\|_p$$

$$(4.4) \quad \|S_k(q)1 - S_k(q_0)1\|_{\mathcal{B}(W^{1,\bar{p}})} \leq c \|q - q_0\|_p$$

which follow from (4.1), (4.2), and linearity in q .

By the second resolvent formula, for q sufficiently close to q_0 , we have

$$R(q) - R(q_0) = R(q_0) (S(q) - S(q_0)) R(q)$$

so that

$$R(q) = [I - R(q_0) (S(q) - S(q_0))]^{-1} R(q_0)$$

These computations can be justified if $\|R(q_0) [S(q) - S(q_0)]\|_{\mathcal{B}(W^{1,\bar{p}})} < 1/2$. From estimate (4.3), we see that it suffices to take

$$\|q - q_0\|_p \leq \left(2c \|R(q_0)\|_{\mathcal{B}(W^{1,\bar{p}})}\right)^{-1}.$$

Thus, taking $r = \left(2c \|R(q_0)\|_{\mathcal{B}(W^{1,\bar{p}})}\right)^{-1}$, we have

$$(4.5) \quad \|R(q)\|_{\mathcal{B}(W^{1,\bar{p}})} \leq 2 \|R(q_0)\|_{\mathcal{B}(W^{1,\bar{p}})}.$$

Using the identity

$$\mu(x, k, q) - \mu(x, k, q') = R(q)S_k(q)1 - R(q_0)S_k(q_0)1,$$

the estimates (4.3), (4.4), (4.5), and the second resolvent formula, we recover the claimed estimate. \square

Now we can prove continuity of \mathbf{t} as a function of q .

Lemma 4.2. *Suppose that $p \in (1, 2)$, $\rho > 1$, that $q \in L^p_\rho(\mathbb{R}^2)$, and $\{q_n\}$ is a sequence from $L^p_\rho(\mathbb{R}^2)$ with $q_n \rightarrow q$ in $L^p_\rho(\mathbb{R}^2)$. Finally, let $\mathbf{t}_n = \mathcal{T}(q_n)$ and $\mathbf{t} = \mathcal{T}(q)$. Then, for all non-exceptional nonzero k , $\mathbf{t}_n(k) \rightarrow \mathbf{t}(k)$ pointwise.*

Proof. Using Lemma 4.1 we estimate

$$\begin{aligned} |\mathbf{t}_n(k) - \mathbf{t}(k)| &\leq \left| \int e_k(x)(q_n(x) - q(x))\mu_n(x) dm(x) \right| \\ &\quad + \left| \int e_k(x)q(x)(\mu_n(x) - \mu(x)) dm(x) \right| \\ &\leq \|q_n - q\|_{L^1} \|\mu_n\|_{L^\infty} + \|q\|_{L^1} \|\mu_n - \mu\|_{L^\infty} \\ &\leq \|q_n - q\|_{L^p_\rho} (\|\mu_n - 1\|_{W^{1,\bar{p}}} + 1) + \|q\|_{L^p_\rho} \|\mu_n - \mu\|_{W^{1,\bar{p}}} \end{aligned}$$

and conclude that $\mathbf{t}_n(k) \rightarrow \mathbf{t}(k)$. \square

4.2. Continuous dependence of reconstructed q on \mathbf{t} . For notational convenience, we combine the time dependence of the scattering data

$$\mathbf{s}(k, \tau) = \exp(i\tau(k^3 + \bar{k}^3))\mathbf{s}(k, 0)$$

with the spatial oscillation $e_{-x}(k)$ into the factor $e^{i\tau S}$ where

$$S(x, k, \tau) = -\frac{kx + \bar{k}\bar{x}}{\tau} + (k^3 + \bar{k}^3).$$

We note the following estimates.

Lemma 4.3. *Suppose that $\mathbf{s} \in L^2(\mathbb{C})$ and $\tilde{r} > 2$. Then $(I - T_{x,\tau})^{-1}$ exists as an operator in $\mathcal{B}(L^{\tilde{r}})$ and*

$$\sup_{(x,\tau) \in \mathbb{R}^2 \times \mathbb{R}} \|(I - T_{x,\tau})^{-1}\|_{\mathcal{B}(L^{\tilde{r}})} < \infty.$$

Proof. We will prove the assertion in four steps. First, we will show that $T_{x,\tau}$ is a compact operator on $L^{\tilde{r}}(\mathbb{C})$ with

$$(4.6) \quad \sup_{(x,\tau) \in \mathbb{R}^2 \times \mathbb{R}} \|T_{x,\tau}\|_{\mathcal{B}(L^{\tilde{r}})} \leq C(\tilde{r}) \|\mathbf{s}\|_{L^2(\mathbb{C})}.$$

Second, we will prove that $\ker(I - T_{x,\tau})$ is trivial, so that, by the Fredholm alternative, the operator $(I - T_{x,\tau})^{-1}$ exists for all $(x, \tau) \in \mathbb{R}^2 \times \mathbb{R}$.

Third, we will show that

$$(4.7) \quad \lim_{T \rightarrow \infty} \sup_{x \in \mathbb{R}^2, |\tau| \geq T} \|T_{x,\tau}^2\|_{\mathcal{B}(L^{\tilde{r}})} = 0$$

and that for each $T > 0$,

$$(4.8) \quad \lim_{R \rightarrow \infty} \sup_{|\tau| \leq T, |x| \geq R} \|T_{x,\tau}^2\|_{\mathcal{B}(L^{\tilde{r}})} = 0.$$

It then follows that there are numbers R and T for which $\|T_{x,\tau}^2\|_{\mathcal{B}(L^{\tilde{r}})} < 1/2$ whenever $|x| \geq R$ or $|\tau| \geq T$, and that therefore

$$(4.9) \quad \|(I - T_{x,\tau}^2)^{-1}\|_{\mathcal{B}(L^{\tilde{r}})} \leq 2$$

on this set. Using the identity $(I - A)^{-1} = (I - A^2)^{-1}(I + A)$, the estimates (4.6), and (4.9), we recover

$$\left\| (I - T_{x,\tau})^{-1} \right\|_{\mathcal{B}(L^{\tilde{r}})} \leq 2 \left(1 + C \|\mathbf{s}\|_{L^2(\mathbb{C})} \right)$$

for all such (x, τ) .

Fourth, we will use a continuity-compactness argument to show that for any positive numbers R and T ,

$$\sup_{|x| \leq R, |\tau| \leq T} \left\| (I - T_{x,\tau})^{-1} \right\|_{\mathcal{B}(L^{\tilde{r}})}$$

is finite.

(1) The norm estimate is immediate from (2.4) with $p = q = \tilde{r}$. To show that $T_{x,\tau}$ is compact, it follows from (4.6) and density that we may take $\mathbf{s} \in C_0^\infty(\mathbb{C})$ without loss. It suffices to prove that the Banach space adjoint $T'_{x,\tau} = e^{i\tau S} \mathbf{s} P_k$ is a compact operator on $L^{\tilde{r}'}(\mathbb{C})$. Let Ω be the support of \mathbf{s} . If $f \in L^{\tilde{r}'}(\mathbb{C})$ then $P_k f \in L^{2\tilde{r}/(\tilde{r}-2)}(\mathbb{C})$ by (2.1) while $\nabla P_k f \in L^{\tilde{r}'}(\mathbb{C})$ by (2.2). Thus

$$\|P_k f\|_{W^{1,\tilde{r}'}(\mathbb{C})} \leq C \left(1 + |\Omega|^{1/2} \right) \|f\|_{\tilde{r}'}$$

and compactness follows from the Rellich-Kondrakov Theorem.

(2) To see that $\ker(I - T_{x,\tau})$ is trivial, note that by Lemma 2.2, $\psi \in \ker(I - T_{x,\tau})$ if and only if $\bar{\partial}\psi = e^{i\tau S} \mathbf{s} \psi$. Since $\mathbf{s} \in L^2(\mathbb{C})$ and $\psi \in L^{\tilde{r}}(\mathbb{C})$, it follows from Theorem 2.3 that $\psi = 0$.

(3) We will prove (4.7) and (4.8) by proving the corresponding estimates for $(T'_{x,\tau})^2$ on $L^{\tilde{r}'}(\mathbb{C})$. First, note that $T'_{x,\tau} = e^{i\tau S} W$ where $W = \mathbf{s} P_k$ is a compact operator independent of (x, τ) . We have the estimate $\|T'_{x,\tau}\|_{\mathcal{B}(L^{\tilde{r}'})} \leq \|W\|_{\mathcal{B}(L^{\tilde{r}'})}$ uniform in (x, τ) . For any $\epsilon > 0$ there is a finite-rank operator F on $L^{\tilde{r}'}(\mathbb{C})$ so that $\|W - F\|_{\mathcal{B}(L^{\tilde{r}'})} < \epsilon$. The operator F is a finite sum $\sum_j \langle \varphi_j, \cdot \rangle \psi_j$ where $\varphi_j \in L^{\tilde{r}}(\mathbb{C})$, $\psi_j \in L^{\tilde{r}'}(\mathbb{C})$, and $\langle \cdot, \cdot \rangle$ is the usual dual pairing of $L^{\tilde{r}}(\mathbb{C})$ and $L^{\tilde{r}'}(\mathbb{C})$. A density argument shows that we may take ψ_j and φ_j in $\mathcal{S}(\mathbb{C})$ without loss. Thus, it suffices to show that

$$(4.10) \quad \lim_{T \rightarrow \infty} \sup_{x \in \mathbb{R}^2, |\tau| \geq T} |\langle \varphi, e^{i\tau S} \psi \rangle| = 0$$

and that, for each fixed $T > 0$,

$$(4.11) \quad \lim_{R \rightarrow \infty} \sup_{|\tau| \leq T, |x| \geq R} |\langle \varphi, e^{i\tau S} \psi \rangle| = 0$$

where φ and ψ belong to $\mathcal{S}(\mathbb{C})$.

Let $f(k) = \psi(k)\varphi(k)$. A short computation shows that, for $S = S(x, k, \tau)$,

$$\langle \varphi, e^{i\tau S} \psi \rangle = \int I_\tau(x - y) (\mathcal{F}^{-1} f)(y) dy$$

where I_τ is given by (2.5) and

$$(\mathcal{F}^{-1} f)(x) = \frac{1}{\pi} \int e_{-k}(x) f(k) dm(k).$$

We now appeal to Lemma 2.5 to estimate

$$(4.12) \quad |\langle \varphi, e^{i\tau S} \psi \rangle| \leq \int \tau^{-2/3} (1 + |(x - y)/\tau|)^{-1/2} |(\mathcal{F}^{-1} f)(y)| dm(y).$$

To prove (4.10), we estimate the right-hand side of (4.12) by $T^{-2/3} \|\mathcal{F}^{-1}\|_{L^1}$. This gives (4.7). To prove (4.11) we make the change of variables $\xi = (x - y)/\tau^{1/3}$ and obtain

$$\begin{aligned} |\langle \varphi, e^{i\tau S} \psi \rangle| &\leq \int (1 + |\xi|)^{-1/2} \left| (\mathcal{F}^{-1} f)(x - \xi \tau^{1/3}) \right| dm(\xi) \\ &\leq C(f, N)(I_1(x, \tau) + I_2(x, \tau)) \end{aligned}$$

where $C(f, N)$ depends on the Schwartz seminorms of f and

$$\begin{aligned} I_1(x, \tau) &= \int_{|\xi| \leq |x|/(2T^{1/3})} (1 + |\xi|)^{-1/2} (1 + |x - \xi \tau^{1/3}|)^{-N} dm(\xi) \\ I_2(x, \tau) &= \int_{|\xi| \geq |x|/(2T^{1/3})} (1 + |\xi|)^{-1/2} (1 + |x - \xi \tau^{1/3}|)^{-N} dm(\xi). \end{aligned}$$

Equation (4.11) now follows from the preceding arguments and the elementary estimates

$$\begin{aligned} |I_1(x, \tau)| &\leq c(N) \left(\frac{|x|}{2T} \right)^2 (1 + |x|)^{-N} \\ |I_2(x, \tau)| &\leq c(N) \left(1 + \frac{|x|}{2T^{1/3}} \right)^{-1/2}. \end{aligned}$$

This implies (4.8).

(4) We claim that the map

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{R} &\longrightarrow \mathcal{B}(L^{\tilde{r}'}(\mathbb{C})) \\ (x, \tau) &\mapsto (I - T_{x, \tau})^{-1} \end{aligned}$$

is continuous. If so, it is continuous on the compact set $\{(x, \tau) : |x| \leq R, |\tau| \leq T\}$ hence bounded there. By the second resolvent formula and duality, it suffices to show that the map

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{R} &\longrightarrow \mathcal{B}(L^{\tilde{r}'}(\mathbb{C})) \\ (x, \tau) &\mapsto T'_{x, \tau} \end{aligned}$$

is continuous. But, viewed as multiplication operators on $L^{\tilde{r}'}(\mathbb{C})$, $e^{i\tau S(\cdot, x', \tau')} \rightarrow e^{i\tau S(\cdot, x, \tau)}$ as $(x', \tau') \rightarrow (x, \tau)$ in the strong operator topology. Since

$$T'_{x', \tau'} - T'_{x, \tau} = \left(e^{i\tau S(\cdot, x', \tau')} - e^{i\tau S(\cdot, x, \tau)} \right) W$$

for a fixed compact operator W , it follows that

$$\|T'_{x', \tau'} - T'_{x, \tau}\|_{\mathcal{B}(L^{\tilde{r}'})} \rightarrow 0 \text{ as } (x', \tau') \rightarrow (x, \tau).$$

This proves the required continuity. \square

Given the uniform resolvent bounds, we can use the second resolvent formula and the continuous dependence of $T_{x, \tau}$ on its parameters to prove various continuity results about the resolvent.

Lemma 4.4. *Suppose that $\mathbf{s} \in L^2(\mathbb{C})$. For any $\tilde{r} > 2$,*

(i) *The mapping*

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{R} &\longrightarrow \mathcal{B}(L^{\tilde{r}}) \\ (x, \tau) &\mapsto (I - T_{x,\tau})^{-1} \end{aligned}$$

is continuous.

(ii) *The mapping*

$$\begin{aligned} L^2(\mathbb{C}) &\longrightarrow C^0(\mathbb{R}^2 \times \mathbb{R}, \mathcal{B}(L^{\tilde{r}})) \\ \mathbf{s} &\mapsto \left((x, \tau) \mapsto (I - T_{x,\tau})^{-1} \right) \end{aligned}$$

is continuous.

Proof. (i) This was already proved in the proof of Lemma 4.3.

(ii) Fix (x, τ) and write $T_{x,\tau} = T_{x,\tau}(\mathbf{s})$ to emphasize the dependence on \mathbf{s} . From the second resolvent formula

$$\begin{aligned} (I - T_{x,\tau}(\mathbf{s}_1))^{-1} - (I - T_{x,\tau}(\mathbf{s}_2))^{-1} \\ = (I - T_{x,\tau}(\mathbf{s}_2))^{-1} [T_{x,\tau}(\mathbf{s}_1) - T_{x,\tau}(\mathbf{s}_2)] (I - T_{x,\tau}(\mathbf{s}_1))^{-1} \end{aligned}$$

and the fact that

$$\sup_{(x,\tau) \in \mathbb{R}^2 \times \mathbb{R}} \|T_{x,\tau}(\mathbf{s}_1) - T_{x,\tau}(\mathbf{s}_2)\|_{\mathcal{B}(L^{\tilde{r}})} \leq C_{\tilde{r}} \|\mathbf{s}_1 - \mathbf{s}_2\|_2$$

we easily deduce that $\left\| (I - T_{x,\tau}(\mathbf{s}))^{-1} \right\|_{\mathcal{B}(L^{\tilde{r}})}$ is bounded uniformly in (x, τ) for \mathbf{s} in a small metric ball in L^2 whose radius depends on the center but is uniform in (x, τ) . For \mathbf{s}_1 and \mathbf{s}_2 in such a metric ball B we may estimate

$$\sup_{(x,\tau) \in \mathbb{R}^2 \times \mathbb{R}} \left\| (I - T_{x,\tau}(\mathbf{s}_1))^{-1} - (I - T_{x,\tau}(\mathbf{s}_2))^{-1} \right\|_{\mathcal{B}(L^{\tilde{r}})} \leq C(B) \|\mathbf{s}_1 - \mathbf{s}_2\|_2.$$

This gives the claimed continuity. \square

We have already established in Lemma (2.9) that, for $\mathbf{s} \in \mathcal{X}_{n,r}^\epsilon$, the derivatives $\partial_\tau^m D_x^\alpha (\mu(x, \cdot, \tau))$ exist in $L^{\tilde{r}}(\mathbb{C})$, provided that $(3m + |\alpha|) \leq n$. We are now ready to prove the continuity of several key maps in \mathbf{s} .

Lemma 4.5. *Given $r \in (1, 2)$*

(i) *For any $n \geq 3m + |\alpha|$, the maps*

$$\begin{aligned} \mathcal{X}_{n,r}^\epsilon &\longrightarrow C(\mathbb{R}^2 \times \mathbb{R}, L^{\tilde{r}}(\mathbb{C})) \\ \mathbf{s} &\mapsto \left((x, \tau) \mapsto \partial_\tau^m D_x^\alpha (\mu(x, \cdot, \tau) - 1) \right) \end{aligned}$$

are continuous.

(ii) *For any $n \geq 3m + |\alpha| + 2$, the map*

$$\begin{aligned} \mathcal{X}_{n,r}^\epsilon &\mapsto C^0(\mathbb{R}^2 \times \mathbb{R}) \\ \mathbf{s} &\mapsto \partial_\tau^m D_x^\alpha q(x, \tau) \end{aligned}$$

is continuous.

Proof. For any $n \geq 3m + |\alpha|$ we will show the map

$$\begin{aligned} \mathcal{X}_{n,r}^\epsilon &\longrightarrow C(\mathbb{R}^2 \times \mathbb{R}, L^{\tilde{r}}(\mathbb{C})) \\ \mathbf{s} &\mapsto \left((x, \tau) \mapsto \partial_\tau^m D_x^\alpha [T_{x,\tau}(\mathbf{s})(1)] \right) \end{aligned}$$

is continuous. The argument is similar to the proof of [19, Lemma 5.4]. From the computation

$$(4.13) \quad \partial_\tau^m D_x^\alpha T_{x,\tau} 1 = P_k \left[e^{itS} (-2ik_2)^{\alpha_1} (2ik_1)^{\alpha_2} [i(k^3 + \bar{k}^3)]^m \mathbf{s}(\cdot) \right]$$

and (2.1), we see that it suffices to bound $\| |\cdot|^\ell (\mathbf{s}_1(\cdot) - \mathbf{s}_2(\cdot)) \|_r$ by $\|\mathbf{s}_1 - \mathbf{s}_2\|_{\mathcal{X}_{n,r}^\epsilon}$, where $\ell = 3m + |\alpha| \leq n$. This is immediate from the definition.

Part (i) is the same as [19, Lemma 5.5] except here we keep track of the continuous dependence on the scattering data. We need to check that the expression (2.9) defines a continuous $C(\mathbb{R}^2 \times \mathbb{R}; L^{\tilde{r}}(\mathbb{C}))$ -valued function of \mathbf{s} . By Lemma 4.4(ii), it suffices to show that $\mathbf{s} \mapsto P_k f$ (with f given by (2.10)) has the same property. A typical term of $P_k f$ takes the form

$$P_k [\mathbf{s}(\cdot) e^{itS} k^\beta (i((\cdot)^3 + (\bar{\cdot})^3))^\ell (\partial_\tau^\ell \partial_x^\beta \mu)(x, \cdot, \tau)]$$

where $|\beta| + \ell \leq m - 1$. Assuming inductively that the derivatives $\partial_\tau^\ell \partial_x^\beta \mu$ are continuous $C(\mathbb{R}^2 \times \mathbb{R}, L^{\tilde{r}}(\mathbb{C}))$ -valued functions of \mathbf{s} , we can now use (2.4) with $q = \tilde{r}$, $p = r$, and $s = 2$ to obtain the required continuity. Thus it remains to prove that $\mu(x, \cdot, k) - 1$ is a continuous $C(\mathbb{R}^2 \times \mathbb{R}; L^{\tilde{r}}(\mathbb{C}))$ -valued function of \mathbf{s} . This is an immediate consequence of Lemma 4.4(ii) and equation (4.13).

Part (ii) follows directly from part (i). The proof is the same as in Music [19, Lemma 5.6]. The result follows from equation (1.9), part (i), and the fact that $\mathbf{s} \in \mathcal{X}_{n+1,r}^\epsilon \subset L_n^1$. \square

5. SYMMETRIES OF THE SCATTERING TRANSFORM

In this section we prove that $q(x)$ is real if and only if $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$. This symmetry was proved in the negative energy case by Grinevich and Manakov in [10] where they also sketch an argument for the symmetry in the zero energy case.

Here we will write this argument directly in the language of the zero-energy case. First, we note that Lemma 4.1 which allows us to prove relevant formulas for smooth potentials and conclude that these formulas continue to hold for $q \in L_\rho^p(\mathbb{R}^2)$ by continuity.

Lemma 5.1. *If $q \in L_\rho^p(\mathbb{R}^2)$ for $p \in (1, 2)$ and $\rho > 2/p'$ is real and k and $-k$ are not exceptional points, then $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$.*

Proof. First assume that $q \in C_c^\infty(\mathbb{R}^2)$ so that the corresponding $\mu(x, k)$ is smooth. Consider the form

$$\omega = e_k \overline{\partial_x \mu}(x, -k) dx + e_k \overline{\partial_x \mu}(x, k) d\bar{x}$$

where

$$dx = dx_1 + idx_2, \quad d\bar{x} = dx_1 - idx_2.$$

We have

$$\begin{aligned} d\omega &= e_k \left((\partial_x + ik) \overline{\partial_x \mu}(x, k) - \overline{(\partial_x - ik) \partial_x \mu}(x, -k) \right) dx \wedge d\bar{x} \\ &= e_k \left(q(x) \mu(x, k) - q(x) \overline{\mu(x, -k)} \right) dx \wedge d\bar{x}. \end{aligned}$$

Hence, by Stokes' Theorem,

$$\mathbf{t}(k) - \overline{\mathbf{t}(-k)} = \int_{\mathbb{R}^2} e_k(x)q(x)\mu(x, k) - e_k(x)q(x)\overline{\mu(x, -k)} dm(x) = 0.$$

To obtain the result for general q , we approximate $q \in L^p_\rho(\mathbb{R}^2)$ by a sequence $\{q_n\}$ from $C_c^\infty(\mathbb{R}^2)$ and appeal to Lemma 4.2. \square

Now, assuming that the scattering data $\mathbf{t}(k)$ has this symmetry, we prove that, if $q(x)$ is computed using the reconstruction formula (1.5), then $q(x) = \overline{q(x)}$. For simplicity, we also assume that $\mathbf{t}(k) = 0$ in a neighborhood of the origin. We follow the proof presented in [9, Theorem 3.4], just rewriting it in terms of the zero energy problem.

Lemma 5.2. *If $\mathbf{t} \in C_c^\infty(\mathbb{C})$ with $\mathbf{t}(k) = 0$ for $|k| < \epsilon$ for some $\epsilon > 0$ and $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$ then $q(x)$ defined by equation (1.5) is real. Moreover, the solution $\mu(x, k)$ of the $\bar{\partial}$ -problem (1.4) satisfies*

$$\bar{\partial}_x(\partial_x + ik)\mu(x, k) = q(x)\mu(x, k)$$

in distribution sense.

Proof. Consider the real differential form

$$\omega = \frac{\mu(x, k)\mu(x, -k)}{k} dk + \frac{\overline{\mu(x, k)\mu(x, -k)}}{\bar{k}} d\bar{k}.$$

Using the symmetry $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$ and (1.4), it is not difficult to see that ω is a closed form. It now follows by Stokes' Theorem applied to the region $R^{-1} \leq |k| \leq R$, the large- k asymptotic behavior of μ and $\bar{\mu}$, and the identities

$$\oint_\gamma \frac{dk}{k} = - \oint_\gamma \frac{d\bar{k}}{\bar{k}} = 2\pi i,$$

true for any simple closed contour γ , that

$$(5.1) \quad \mu(x, 0)^2 = \left[\overline{\mu(x, 0)} \right]^2.$$

Thus, $\mu(x, 0)$ is either purely real or purely imaginary. Next consider the differential operators

$$\begin{aligned} P_1\psi &= -\bar{\partial}_x(\partial_x + ik)\psi + q\psi \\ P_2\psi &= -\partial_x(\bar{\partial}_x - i\bar{k})\psi + q\psi \end{aligned}$$

where q is defined by the expansion (2.11) to be $q = i\bar{\partial}_x a_1$. Let $\chi_1 = P_1\mu$ and $\chi_2 = P_2\bar{\mu}$. We need to show $\chi_1 = \chi_2 = 0$. From the expansion (2.11), we have

$$\lim_{|k| \rightarrow \infty} \chi_1(x, k) = 0.$$

By (5.1), we have

$$(5.2) \quad \chi_1(x, 0)^2 - \chi_2(x, 0)^2 = 0,$$

and χ_1 and χ_2 satisfy

$$\begin{aligned} (\bar{\partial}_k \chi_1)(x, k) &= e_{-x}(k) \mathbf{s}(k) \chi_2(x, k) \\ (\partial_k \chi_2)(x, k) &= e_x(k) \overline{\mathbf{s}(k)} \chi_1(x, k). \end{aligned}$$

where, for each fixed x ,

$$\lim_{|k| \rightarrow \infty} \chi_2(x, k) = \lim_{|k| \rightarrow \infty} i\partial_x \bar{a}_1 + q + O(|k|^{-1}) = i\partial_x \bar{a}_1(x) + q(x).$$

It is not (yet) clear that $i\partial_x \bar{a}_1(x) + q(x) = 0$ but we will prove this using the condition (5.2).

To this end, consider the one-form

$$\eta = \frac{\chi_1(k, z)\chi_1(-k, z)}{k} dk + \frac{\chi_2(k, z)\chi_2(-k, z)}{k} d\bar{k}.$$

A computation analogous to the one for ω together with (5.2) shows that η is a closed form and that

$$\chi_1(x, \infty)^2 - \chi_2(x, \infty)^2 = \chi_1(x, 0)^2 - \chi_2(x, 0)^2$$

where $\chi_i(x, \infty)$ means $\lim_{|k| \rightarrow \infty} \chi_i(x, k)$ for $i = 1, 2$. By (5.2), the right-hand side is zero, and also $\chi_1(x, \infty) = 0$. We may conclude then that $\chi_2(x, \infty) = 0$, thus $q = i\partial_x \bar{a}_1 = \bar{q}$ as desired. \square

Combining the above result with Lemma 4.5 we have:

Corollary 5.3. *If $\mathbf{s} \in \mathcal{X}_{2,r}^\epsilon(\mathbb{C})$ for $r \in (1, 2)$ and $\epsilon > 0$ then $q(x)$ defined by equation (1.5) is real. Moreover, the solution $\mu(x, k)$ of the $\bar{\partial}$ -problem (1.4) satisfies*

$$\bar{\partial}_x(\partial_x + ik)\mu(x, k) = q(x)\mu(x, k)$$

in the sense of distributions.

It is now simple to prove the interesting fact that all scattering transforms in $\mathcal{X}_{n,r}^\epsilon$ come from critical or subcritical potentials.

Proposition 5.4. *If $\mathbf{s} \in \mathcal{X}_{2,r}^\epsilon$ then $q(x) = [\mathcal{Q}\mathbf{t}](x)$ is critical or subcritical.*

Proof. This result follows easily using an approximation argument. Define $q_\epsilon(x) = \mathcal{Q}(\mathbf{t}_\epsilon)(x)$ where $\mathbf{t}_\epsilon(k) = \chi_\epsilon(k)\mathbf{t}(k)$ and χ_ϵ is a smooth radial function which is 1 for $|k| > \epsilon$ and 0 for $|k| < \epsilon/2$. Nachman [21, Theorem 4.1] proves that the solutions $\mu_\epsilon(x, k)$ to equation (1.8) satisfy $\inf_{x,k} |\mu_\epsilon(x, k)| > 0$ and from (5.1) $\mu_\epsilon(x, 0)$ is either real or imaginary. By multiplying this by the appropriate constant and using Corollary 5.3, we have $c\mu_\epsilon(k, 0)$ is a positive solution to the Schrödinger equation with potential q_ϵ . By Lemma 4.5 $q_\epsilon \in C^0(\mathbb{R}^2)$, so by [7, Theorem 2.12] q_ϵ is critical or subcritical. By Lemma 4.5, on any compact subset $\Omega \subset \mathbb{R}^2$ we have $\lim_{\epsilon \rightarrow 0} |q_\epsilon(x) - q(x)| = 0$. By Definition 1.1, a potential q is either critical or subcritical if the associated quadratic form (1.10) is positive. That is, for every q_ϵ and any $\psi \in C_c^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} |\partial_x \psi|^2 + q_\epsilon |\psi|^2 dm(x) \geq 0.$$

Taking limits preserves the non-negativity and the result follows. \square

The following identities are a consequence of $\mu(x, k, \tau)$ solving

$$\bar{\partial}_x(\partial_x + ik)\mu = q\mu$$

point-wise as a map $k \rightarrow L^r$ and the large- k expansion (2.11).

Corollary 5.5. *Assume $\mathbf{s} \in \mathcal{X}_{n+1,r}^\epsilon$ and define $q = i\bar{\partial}_x a_1$. Then, the following identities hold:*

$$(5.3) \quad i\bar{\partial}_x a_n = -\bar{\partial}_x \partial_x a_{n-1} + (i\bar{\partial}_x a_1) a_{n-1},$$

which for $n = 2$ simplifies to

$$(5.4) \quad i\bar{\partial}_x a_2 = \bar{\partial}_x \left(-\partial_x a_1 + i\frac{a_1^2}{2} \right).$$

Additionally we have

$$(5.5) \quad i\partial_x a_2 = \partial_x \left(-\partial_x a_1 + i\frac{a_1^2}{2} \right).$$

Proof. Identity (5.3) follows by plugging in the large- k expansion (2.11) into (1.8). We get identity (5.5) by applying Liouville's theorem to the analytic function $ia_2 - \partial_x a_1 + i\frac{a_1^2}{2}$ and noting that a_1 , a_2 , and $\partial_x a_1$ are bounded. \square

6. SOLVING THE NOVIKOV-VESELOV EQUATION

In this section, we show that $q(x, \tau)$ defined by (1.9) solves the NV equation at zero energy. We will assume $\mathbf{s} \in \mathcal{X}_{5,r}^\epsilon$. Note that, for any \mathbf{s} of this form, the function $q(x, \tau)$ defined by (1.9) is real by Lemma 5.3 and the fact that the map $\mathbf{t} \mapsto e^{i\tau(k^3 + \bar{k}^3)} \mathbf{t}$ preserves the symmetry $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$ and the norms. For such transforms, we will prove that $q(x, \tau)$ does indeed solve the NV equation. It will then follow from Lemma 2.6 that the inverse scattering method produces a global solution for critical or subcritical initial data q_0 with $q_0 \in W_{\rho}^{5,p}(\mathbb{R}^2)$ for some $p \in (1, 2)$, and $\rho > 1$.

This section exploits ideas from Section 3 of the paper of Croke, Mueller, Music, Perry, Siltanen, and Stahel [6], where the Manakov triple representation of the NV equation [17] is used to study the inverse scattering method. In order to show that $q(x, \tau)$ as defined by (1.9) in fact solves the NV equation, we need an equation of motion for $\mu(x, k, \tau)$.

We will show directly from the $\bar{\partial}_k$ -equation for μ that $\mu(x, k, \tau)$ satisfies an explicit equation of motion. It then becomes a matter of careful computation to show that the time derivative of (1.9) satisfies the NV equation.

Lemma 6.1. *If $\mathbf{s} \in \mathcal{X}_{4,r}^\epsilon$ for $r \in (1, 2)$ and $\epsilon > 0$ then the function $\mu(x, k, \tau)$ defined by (1.8) satisfies the equation*

$$(\partial_\tau - ik^3)\mu = \left[\bar{\partial}_x^3 + (\partial_x + ik)^3 - 3u(\partial_x + ik) - 3\bar{u}\bar{\partial}_x \right] \mu(x, k)$$

where $u = i\partial_x a_1$.

Proof. We have the two equations

$$\bar{\partial}_x(\partial_x + ik)\mu(x, k) = q(x)\mu(x, k)$$

and

$$\bar{\partial}_k \mu(x, k, \tau) = \mathbf{s}(k, \tau) e_{-x}(k) \bar{\mu}$$

Expanding $\mu(x, k, \tau)$ up to order k^3 , we have

$$(6.1) \quad \mu(x, k, \tau) = 1 + \frac{a_1(x, \tau)}{k} + \frac{a_2(x, \tau)}{k^2} + \frac{a_3(x, \tau)}{k^3} + o(k^{-3}).$$

By commuting the following specially chosen differential operators through the $\bar{\partial}_k$ equation, we obtain the three identities

$$\begin{aligned}\bar{\partial}_k(\partial_\tau - ik^3)\mu &= \mathbf{s}(k, \tau)e_{-x}(k)\overline{(\partial_\tau - ik^3)\mu}, \\ \bar{\partial}_k[\bar{\partial}_x^3 + (\partial_x + ik)^3]\mu &= \mathbf{s}(k, \tau)e_{-x}(k)\overline{[\bar{\partial}_x^3 + (\partial_x + ik)^3]\mu},\end{aligned}$$

and

$$\bar{\partial}_k[-3\bar{u}\bar{\partial}_x - 3u(\partial_x + ik)]\mu = \mathbf{s}(k, \tau)e_{-x}(k)\overline{[-3\bar{u}\bar{\partial}_x - 3u(\partial_x + ik)]\mu}.$$

We can combine these in such a way to get a formula for $\partial_\tau\mu$. Adding the three identities we conclude that

$$\Psi = \left[\partial_\tau - (\bar{\partial}_x^3 + \partial_x^3 + 3ik\partial_x^2 - 3k^2\partial_x - 3u(\partial_x + ik) - 3\bar{u}\bar{\partial}_x) \right] \mu$$

satisfies

$$\bar{\partial}_k\Psi = \mathbf{s}(k, \tau)e_{-x}(k)\bar{\Psi}.$$

We will prove that $\Psi = O(k^{-1})$ so that Theorem 2.3 implies that $\Psi \equiv 0$. Plugging (6.1) into the expression for Ψ , it is easy to see that we get no terms of order k^2 or higher. Collecting all terms of order k^1 we find

$$-3k\partial_x a_1 - 3iku = 0$$

which is zero by our choice of u . Collecting terms of order k^0 gives us

$$(6.2) \quad 3i\partial_x^2 a_1 - 3\partial_x a_1 - 3iua_1 = 3i\partial_x \left[\partial_x a_1 + ia_2 - \frac{i}{2}a_1^2 \right].$$

which is zero by identity (5.5). Thus, the order k^0 terms in equation (6.2) are zero. This finishes the proof that $\Psi \equiv 0$. \square

We are now ready to prove that $q(x, \tau) = i\bar{\partial}_x a_1(x, \tau)$ solves the Novikov-Veselov equation. This follows by looking at the terms of order k^{-1} after plugging in the asymptotic expansion into the evolution equation for μ and using the identities from Corollary 5.5.

Corollary 6.2. *If $\mathbf{s} \in \mathcal{X}_{5,r}^\epsilon$ then the function $q(x, \tau)$ satisfies*

$$\partial_\tau q = \bar{\partial}_x^3 q + \partial_x^3 q - 3\partial_x(uq) - 3\bar{\partial}_x(\bar{u}q).$$

Proof. We expand Ψ at order k^{-1} to get

$$\partial_\tau a_1 = \bar{\partial}_x^3 a_1 + \partial_x^3 a_1 + 3i\partial_x^2 a_2 - 3\partial_x a_3 - 3u\partial_x a_1 - 3iua_2 - 3\bar{u}\bar{\partial}_x a_1.$$

Applying the operator $i\bar{\partial}_x$ to both sides we get

$$\partial_\tau q = \bar{\partial}_x^3 q + \partial_x^3 q - 3\partial_x^2 \bar{\partial}_x a_2 - 3i\partial_x \bar{\partial}_x a_3 - 3\bar{\partial}_x(u^2) + 3\bar{\partial}_x(ua_2) - 3\bar{\partial}_x(\bar{u}q).$$

Now we use equation (5.3) with $n = 2$ to get

$$\partial_\tau q = \bar{\partial}_x^3 q + \partial_x^3 q - 3\partial_x^2 \bar{\partial}_x a_2 + 3\partial_x^2 \bar{\partial}_x a_2 - 3\partial_x(qa_2) - 3\bar{\partial}_x(u^2) + 3\bar{\partial}_x(ua_2) - 3\bar{\partial}_x(\bar{u}q).$$

$$\partial_\tau q = \bar{\partial}_x^3 q + \partial_x^3 q - 3q\partial_x a_2 - 3\bar{\partial}_x(u^2) + 3u\bar{\partial}_x a_2 - 3\bar{\partial}_x(\bar{u}q).$$

The final step is to show

$$-3q\partial_x a_2 + 3u\bar{\partial}_x a_2 - 3\bar{\partial}_x(u^2) = -3\partial_x(uq).$$

We use equations (5.4) and (5.5) to get

$$\begin{aligned}
 3iq\partial_x \left(-\partial_x a_1 + i\frac{a_1^2}{2} \right) - 3iu\bar{\partial}_x \left(-\partial_x a_1 + i\frac{a_1^2}{2} \right) - 3\bar{\partial}_x(u^2) \\
 = -3q\partial_x u + 3u\partial_x q - 6u\bar{\partial}_x u \\
 = -3q\partial_x u - 3u\partial_x q \\
 = -3\partial_x(uq)
 \end{aligned}$$

□

The main result of our paper now follows trivially.

Proof of Theorem 1.2. By Lemma 2.6 we have $\mathbf{s} \in \mathcal{X}_{5,r}^\epsilon$ when $q_0 \in W_\rho^{5,p}(\mathbb{R}^2)$ for $p \in (1, 2)$, $\rho > 1$, $\epsilon > 0$, and $r \in (\tilde{p}', \infty)$. By Corollary 6.2, $q(x, \tau)$ solves the Novikov-Veselov equation. By Lemma 4.5 part (iii) $q(\cdot, \tau) \in C(\mathbb{R}^2 \times \mathbb{R})$ and hence $q(x, \tau) \rightarrow q(x, 0)$, where $q(x, 0)$, the inverse scattering transform of $\mathbf{s}(k, 0)$, is the initial datum. □

7. CONCLUSION

We have shown that we can solve the Novikov-Veselov equation via inverse scattering for a wide range of potentials. The most significant problem left in zero-energy inverse scattering is the handling of supercritical initial data, for which the exceptional set need not be empty and the scattering transform may have non-integrable singularities of codimension one (see, for example [20]). In addition, Taimanov and Tsarev [26, §4] have constructed supercritical initial data for the NV equation corresponding to a solution that blows up in finite time. This solution currently lacks in interpretation in terms of inverse scattering, and such an interpretation would be of considerable interest.

Another open problem concerns the decay of solutions with subcritical initial data. Lassas, Mueller, and Siltanen [15] prove that certain reconstructed critical potentials have the decay $|(Q\mathbf{t})(x)| \leq C\langle x \rangle^{-2}$. We would like to know that this result, or a similar one, holds for subcritical potentials as well, but the singularity at $k = 0$ in the scattering transform poses additional difficulties.

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